# Turbulence Model for Rotating Flows

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**A second-moment turbulence closure model of the type widely used in modeling geophysical flows is extended to include the effects of rotation on turbulence. The model is the product of a systematic scaling analysis of the Reynolds stress equations in terms of deviations from local isotropy. Both spanwise and axial rotation cases are considered. The model is shown to be conceptually equivalent to, but more self-consistent than, the turbulence models widely used in the simulation of flows in turbomachinery. It is also shown that rotation imparts tensorial properties to eddy viscosity. In the limit of analytically tractable local equilibrium approximation, interesting effects such as complete suppression of turbulence are readily revealed. It is shown that at sufficiently strong destabilizing spanwise rotation, destabilization gives way to restabilization, leading to eventual suppression of turbulence. Also Monin-Obukhov-type similarity theory is applied to the constant flux region of rotating turbulent boundary layers.**

## I. Introduction

FLOWS in turbines, compressors, and other turbomachinery are strongly affected by rotational effects and the effects of flow curvature. These effects may change both the mean flow and the turbulence structure in these devices because of the "extra rate of strains" (see Ref. 1). Shear flows with extra strains such as these are classified as complex shear flows (Lakshminarayana[2]) as opposed to simple shear flows such as a conventional flat plate, zero pressure gradient boundary layer. The influence of extra strain rates on the turbulence structure can be profound, and it is a challenge to model complex turbulent shear flows. Although simple shear flows have been modeled successfully by techniques such as second-moment closure, their success in modeling complex shear flows is not assured a priori.

Rotational terms cause redistribution of turbulence energy among its components and affect the Reynolds stresses modifying the mean flow implicitly. In addition, Coriolis terms explicitly enter the mean momentum equations. The effect of rotation on turbulent energy may be quite dramatic, although Coriolis force is a conservative body force and, therefore, does not enter the energy equation directly.

The effect of rotation on turbulent flows in both geophysics and engineering devices has been a subject of intensive research for the last several decades. A recent review of the state-of-the-art in modeling turbulence in turbomachinery has been given by Lakshminarayana.[2] The models considered in his review are based on turbulence closure schemes developed by Launder et al.[3] and Rodi[4] (models of LRR type hereafter). Rodi[5] has provided an updated overview of these models emphasizing geophysical applications. Closure models other than those belonging to the LRR family have not found wide use in engineering applications. The purpose of the present work is to extend a different family of turbulence closure models first outlined by Mellor and Herring[6] to include the effects of rotation. The models of the Mellor family are based on Rotta and Kolmogorov closure hypotheses and are derived in a self-consistent manner, based on an order-of-magnitude analysis of the Reynolds stress equation in terms of departure from local isotropy (Mellor and Yamada[7]). This approach establishes a hierarchy of turbulence closure models with varying levels of complexity. Two of these models, called the Level 2 and the Level 2½ models, are widely used in geophysical applications (for an update on Mellor family of models and their performance in simulations of various flows, see the review by Mellor and Yamada[8] where additional references can be found). Recently, Galperin et al.,[9] using similar scaling analysis, have proposed a so-called Quasi-Equilibrium Turbulence Energy model (QETE model hereafter) that is the simplest among those models in the Mellor and Yamada hierarchy, which possess a prognostic equation for turbulence energy. That paper also contains some results of simulations of nonrotating turbulent flows using QETE model and comparisons with other Mellor and Yamada models. In a companion study by Galperin et al.,[10] QETE model was applied to geophysical problems with rotation and density stratification. Kantha et al.[11] have applied the local equilibrium version of the Mellor family of models (Level 2) to the general case of rotation and stratification.

As in all second-moment closure models, the empirical constants in the Mellor family of models are regarded as universal constants and, once evaluated (preselected) by whatever means, are kept unchanged from application to application. The constants are usually chosen by appealing to laboratory experiments on simple shear flows. The same constants then are used in model applications, even when additional strains are introduced as in complex shear flows. The empirical constants might be re-evaluated from time to time as more and better experimental data become available, but the important point is that after re-evaluation they are kept unchanged from application to application to ensure universality of the model, because any "fine tuning" of these presumably invariant constants detracts from the attractiveness of the approach.

The main difference between Mellor and LRR families of models is the term in the Reynolds stress equation that originates from pressure-strain covariance and that describes the interaction between turbulence and mean shear—the so-called rapid distortion term. The Mellor models, including QETE, following Rotta,[12] assume this term to be proportional to a product of the isotropic part of the Reynolds stress tensor, $q^2$ ($q^2/2 = k$ is turbulence kinetic energy), and the mean rate of strain tensor. Moreover, the rapid distortion effect is made less important by the use of a small numerical coefficient. LRR model, however, uses a more complicated expression assuming proportionality to a tensor product of a fourth-order rank tensor constructed from the Reynolds stress and second-order rank tensor of mean rate of strain. [The models developed by Wyngaard et al.,[13] So,[14] and at Aeronautical

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Research Associates of Princeton, Inc., (ARAP) (Lewellen[15]) ignore this term altogether and, therefore, do not consider rapid distortion effects. However, this is, strictly speaking, less desirable, leading to an exaggerated role of the return-to-isotropy term (Gibson and Younis[16]).] The LRR approach was conceived to simulate more accurately the evolution of rapidly distorted turbulent flows. Data obtained from experiments on rapid distortion were used to calibrate the empirical constants. However, problems were found in the performance of LRR models when used to simulate complex flows without strong distortion. In the recent work by Gibson and Younis,[16] these difficulties were attributed to the deficiences related to the influence of the strong rapid distortion term and, therefore, the empirical constants were re-evaluated so as to reduce its contribution. Support for a weaker contribution of rapid distortion component on pressure-strain covariance, inherent in Mellor family, has been given by Lumley[17] who pointed out that the rapid distortion problem cannot be simulated by the second-order turbulence closure models assuming local isotropy of the small scales. This is because the imposed mean strain in general dominates the entire spectrum, including the dissipation scales, not just the large energy containing scales. Gibson and Younis[16] reported improved agreement of their revised model with data for simulations of swirling jets. Turbulence models used in turbomachinery, however, have not yet used the revised constants (for examples see Lakshminarayana[2]), which in general might yield better results and to some extent obviate the need for resetting a posteriori the coefficients in the final expressions as discussed below. Such resetting would imply fine tuning of basic constants and, therefore, would be contrary to the basic precept of second-moment closure, namely the invariance of the empirical constants.

The revised LRR model has become much closer conceptually to the Mellor family of models including QETE. For example, the constant in the return-to-isotropy term in the pressure-strain covariance is practically the same. This suggests that the QETE model could be used for complex flows important in engineering with performance no worse and, in fact, better than that of models derived from LRR family, as will be clear below. It is also inherently self-consistent and simple.

## II. Description of the Model

The QETE model describing neutral turbulent flows with rotation is given by the following set of equations in general tensorial form (see Galperin et al.[9] for derivation):

Continuity equation:

$$\frac{\partial U_i}{\partial x_i} = 0 \tag{1}$$

Momentum equation:

$$\frac{\mathrm{D}U_i}{\mathrm{D}t} + \epsilon_{ikl} f_k U_l = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_l}(-\overline{u_i u_l}) \tag{2}$$

where $\mathrm{D}(\cdot)/\mathrm{D}t$ denotes the material derivative,

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x_i}$$

Reynolds stress equation:

$$\overline{u_i u_j} = \frac{\delta_{ij}}{3} q^2 - \frac{3\ell_1}{q}\left[\overline{u_k u_i}\frac{\partial U_j}{\partial x_k} + \overline{u_k u_j}\frac{\partial U_i}{\partial x_k} - C_1 q^2\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right.$$
$$\left. + f_k(\epsilon_{ikl}\overline{u_l u_j} + \epsilon_{jkl}\overline{u_l u_i}) + \frac{2q^3}{3\Lambda_1}\delta_{ij}\right] \tag{3}$$

Turbulence energy equation:

$$\frac{\mathrm{D}q^2}{\mathrm{D}t} - \frac{\partial}{\partial x_k}\left(q\ell S_q \frac{\partial q^2}{\partial x_k}\right) = -2\,\overline{u_k u_l}\frac{\partial U_k}{\partial x_l} - 2\frac{q^3}{\Lambda_1} \tag{4}$$

where $U_i$ and $u_i$ are mean and fluctuating velocities, $P$ is mean pressure, which may be modified by centrifugal force as in Johnston et al.,[18] $f_k = -2\Omega_k$ is the Coriolis vector, where $\Omega_k$ is the vector of angular velocity of the frame of reference $q^2 = \overline{u_i^2}$, and summation convention on indices is implied. Various length scales of turbulence are related to the master length scale $\ell$ (Mellor and Yamada[8]):

$$(\ell_1, \Lambda_1) \equiv (A_1, B_1)\,\ell \tag{5}$$

where

$$A_1 = 0.92, \qquad B_1 = 16.6 \tag{6}$$

It can be shown (Mellor[19]) that the remaining constant $C_1$ is given by

$$C_1 = \tfrac{1}{3}(1 - 6A_1B_1^{-1} - A_1^{-1}B_1^{-1/3}) = 0.08 \tag{7}$$

The constant $S_q$ $(=0.2)$ is the transport coefficient for turbulence energy (Mellor and Yamada[8]). It should be mentioned that in early papers slightly different values of these empirical constants were used (see Mellor and Yamada[7]). Later, when more and better quality experimental data for simple turbulent flows became available, these basic constants were re-evaluated to better fit the new data (Mellor and Yamada[8]). These new values have been kept invariant in all subsequent applications.

The master length scale $\ell$ is described by the $q^2\ell$ equation (Mellor and Yamada[8]):

$$\frac{\mathrm{D}q^2\ell}{\mathrm{D}t} - \frac{\partial}{\partial x_k}\left[q\ell S_q\frac{\partial}{\partial x_k}(q^2\ell)\right] = \ell E_1\left(-\overline{u_k u_l}\frac{\partial U_k}{\partial x_l}\right)$$
$$-\frac{q^3}{B_1}\left[1 + E_2\left(\frac{\ell}{\kappa L}\right)^2\right] \tag{8}$$

where $E_1 = 1.8$, $E_2 = 1.33$, $L$ is a measure of the distance away from the wall, and $\kappa$ is the von Kármán constant $\kappa = 0.4$. Equation (3) contains explicit Coriolis term proportional to $f_k$ describing "extra strains" introduced by rotation. This equation has been obtained by the use of the Rotta hypothesis as applied to a nonrotating system. The Rotta hypothesis can be extended to include effects of rotation of pressure-strain correlation, resulting in additional terms known as "implicit" Coriolis terms in the Reynolds stress equation (see Zeman and Tennekes[20]). However, following Galperin et al.,[10] we shall neglect these terms in this paper. The form of the implicit terms is the same as that of the explicit terms. It is thought that the implicit terms tend to counteract the explicit ones and, therefore, neglecting the implicit terms leads to a slight overestimation of the rotational effects.

If this model is applied to a horizontally homogeneous rotating flow with axis $y$ directed vertically, the momentum and Reynolds stress equations will take the form

$$\frac{\partial U}{\partial t} - f_y W = \frac{\partial}{\partial y}(-\overline{uv}) \tag{9}$$

$$\frac{\partial W}{\partial t} + f_y U = \frac{\partial}{\partial y}(-\overline{vw}) \tag{10}$$

$$\begin{pmatrix} \overline{uv} \\ \overline{vw} \\ \overline{uw} \end{pmatrix} = -\frac{3\ell_1}{q}\begin{pmatrix} (\overline{v^2} - C_1 q^2)\dfrac{\partial U}{\partial y} \\ (\overline{v^2} - C_1 q^2)\dfrac{\partial W}{\partial y} \\ \overline{uv}\dfrac{\partial W}{\partial y} + \overline{vw}\dfrac{\partial U}{\partial y} \end{pmatrix}$$
$$-\frac{3\ell_1}{q}\begin{pmatrix} -f_y\,\overline{vw} + f_z(\overline{v^2} - \overline{u^2}) \\ -f_z\,\overline{uw} + f_y\,\overline{uv} \\ f_z\,\overline{uv} + f_y(\overline{u^2} - \overline{w^2}) \end{pmatrix} \tag{11}$$

$$\begin{bmatrix} \overline{u^2} \\ \overline{v^2} \\ \overline{w^2} \end{bmatrix} = \frac{q^2}{3}\left(1 - \frac{6A_1}{B_1}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3\ell_1}{q}\begin{bmatrix} 2\,\overline{uv}\,\dfrac{\partial U}{\partial y} \\ 0 \\ 2\,\overline{vw}\,\dfrac{\partial W}{\partial y} \end{bmatrix}$$

$$-\frac{6\ell_1}{q}\begin{bmatrix} f_z\,\overline{uv} - f_y\,\overline{uw} \\ -f_z\,\overline{uv} \\ f_y\,\overline{uw} \end{bmatrix} \tag{12}$$

Here, the mean velocity vector is $V = (U, V, W)$, and $f = (0, f_y, f_z)$ is the Coriolis vector. The case $f = (0, f_y, 0)$ corresponds to axial rotation, and $f = (0, 0, f_z)$ represents spanwise rotation. Equations (11) and (12) can be used to relate the Reynolds stress to the mean shear or, in other words, to establish eddy viscosity formulation for the rotational flows under consideration.

The turbulence closure model presented above consists of an algebraic Reynolds stress equation combined with two prognostic equations predicting $q^2$ and $q^2\ell$. Lakshminarayana[2] provided an extensive survey of LRR-type turbulence models developed to simulate complex flows. He noted that the most successful schemes are those combining algebraic Reynolds stress equations with $k$-$\epsilon$ equations, $\epsilon$ being the rate of turbulence dissipation. These models are conceptually similar to the QETE model already presented. However, we believe that the use of the $q^2\ell$ equation is preferable to that of the $\epsilon$ equation since, as shown by Mellor,[21] this equation, as generally derived, is not an equation that deals with a dissipation macroscale. The models reviewed by Lakshminarayana[2] appear to be somewhat less systematic, as they involve additional semi-empirical assumptions relating the material derivative to production, and their coefficients sometimes appear to require a posteriori resetting.

## III. Case of Axial Rotation

In this situation, Eqs. (11) and (12) show that the Reynolds stress is not aligned with the mean strain giving raise to a tensorial eddy viscosity:

$$-\overline{uv} = q\ell\left(S_{MUU}\frac{\partial U}{\partial y} + S_{MUW}\frac{\partial W}{\partial y}\right) \tag{13}$$

$$-\overline{vw} = q\ell\left(S_{MWU}\frac{\partial U}{\partial y} + S_{MWW}\frac{\partial W}{\partial y}\right) \tag{14}$$

where

$$S_{MUU} = B_1^{-1/3}/(1 + 9A_1^2\, Ro_y^{-2}) \tag{15}$$

$$S_{MUW} = 3A_1B_1^{-1/3}Ro_y^{-1}/(1 + 9A_1^2\, Ro_y^{-2}) \tag{16}$$

$$S_{MWU} = -S_{MUW} \tag{17}$$

$$S_{MWW} = S_{MUU} \tag{18}$$

and where $Ro_y$ is a local Rossby number

$$Ro_y \equiv q/\ell f_y \tag{19}$$

One can see that the matrix of $S_M$ is antisymmetric, its denominator does not depend on the sign of $Ro_y$, and it does not explicitly depend on the mean shear. Nonzero off-diagonal elements describe intercomponent mean momentum exchange due to rotation. Increase in rotation decreases vertical turbulent transport. The flow is affected by rotation in a local sense, which is in agreement with conclusions of Lakshminarayana[2] and is supported by the experiments of Hopfinger et al.[22] who reported that in grid-generated turbulence, near-grid region was only slightly affected by rotation. The influence of rotation increased with increasing distance away from the grid. When $Ro_y$ reached the critical value of about 0.2, a transition from three- to two-dimensional turbulence took place. If the value $Ro_y = 0.2$ is substituted into Eq. (15), one finds that $S_{MUU}$ is drastically reduced compared to the nonrotational case, in agreement with experiments of Hopfinger et al. on suppression of three-dimensional turbulence.

The horizontal momentum equations for the class of flows under consideration admit a complex form with a complex eddy viscosity:

$$\frac{\partial \hat{V}}{\partial t} + if_y\,\hat{V} = \frac{\partial}{\partial y}\left(q\ell\hat{S}_M\frac{\partial \hat{V}}{\partial y}\right) \tag{20}$$

where

$$\hat{S}_M = S_{MUU} - iS_{MUW} \tag{21}$$

$$\hat{V} = U + iW \tag{22}$$

Further analysis of Eqs. (13–18) is possible if the local equilibrium assumption is made, allowing the tendency and convection terms in the energy equation, Eq. (4), to be neglected such that

$$-\overline{uv}\frac{\partial U}{\partial y} - \overline{vw}\frac{\partial W}{\partial y} = \frac{q^3}{B_1\ell} \tag{23}$$

Substituting Eqs. (13) and (14) into Eq. (23) yields

$$S_{MUU}\,G^2 = B_1^{-1}(q/\ell)^2 \tag{24}$$

where

$$G^2 = \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 \tag{25}$$

Also, using definition, Eq. (19), one can show that

$$Ro_y^{-1} = (B_1S_{MUU})^{-1/2}\, f_y/|G| \tag{26}$$

where the ratio $f_y/|G|$ can be identified as the axial rotation stability parameter:

$$Rr_y \equiv f_y/|G| \tag{27}$$

Using Eqs. (24–27), the eddy viscosity matrix elements $S_{MUU}$ and $S_{MUW}$ [Eqs. (15 and 16)] can be written as

$$S_{MUU} = B_1^{-1/3}(1 - 9A_1^2B_1^{-2/3}Rr_y^2) \tag{28}$$

$$S_{MUW} = 3A_1B_1^{-1/2}S_{MUU}^{1/2}Rr_y$$
$$= 3A_1B_1^{-2/3}Rr_y(1 - 9A_1^2B_1^{-2/3}Rr_y^2)^{1/2} \tag{29}$$

According to Eq. (24), $S_{MUU} \geq 0$, so that

$$|Rr_y| \leq (9A_1^2B_1^{-2/3})^{-1/2} = 0.923$$

or

$$-0.923 \leq Rr_y \leq 0.923 \tag{30}$$

Vertical turbulent mixing is most intensive in flows without rotation $Rr_y = 0$, which implies also $S_{MUW} = 0$. Increasing rotation monotonically decreases turbulent exchange measured by $S_{MUU}$, whereas the intercomponent exchange measured by $S_{MUW}$ reaches its maximum at $Rr_y = \pm 0.654$. At $Rr_y = \pm 0.923$, both $S_{MUU}$ and $S_{MUW}$ become equal to zero, indicating extinction of three-dimensional turbulence, and in this sense the axial rotation stability parameter $Rr_y$ is similar to the Richardson number for density stratified flows. However, there is also an inherent difference between the stratification and rotational stability numbers: whereas the former was obtained from the turbulence energy equation reflecting the relative importance of mechanical production and buoyancy destruction, the latter describes redistribution of Reynolds stresses and has nothing to do with the energy equation. The authors are not aware of the existence of any observational data about the inequality Eq. (30) at the present time.

Using Eqs. (13), (14), (24), (28), and (29), the following expressions for the absolute value of the nondimensional Reynolds stress can be obtained:

$$\left[\left(\frac{-\overline{uv}}{q^2}\right)^2 + \left(\frac{-\overline{vw}}{q^2}\right)^2\right]^{1/2} = \frac{\ell}{q} |G| (S_{MUU}^2 + S_{MUW}^2)^{1/2}$$
$$= \ell/q \; |G| B_1^{-1/3}(1 - 9A_1^2 B_1^{-2/3} Rr_y^2)^{1/2}$$
$$= \ell/q \; |G| B_1^{-1/6} S_{MUU}^{1/2} = B_1^{-2/3} \tag{31}$$

The last equation is rather interesting, showing that

$$[(-\overline{uv})^2 + (-\overline{vw})^2]^{1/2} = B_1^{-2/3} q^2 \tag{32}$$

Finally, the angle of misalignment $\theta$ between the directions of Reynolds stress and mean shear is given by

$$\tan\theta = 3A_1 B_1^{-1/2} Rr_y S_{MUU}^{-1/2} = 3A_1 B_1^{-1/3} Rr_y (1 - 9A_1^2 B_1^{-2/3} Rr_y^2)^{-1/2} \tag{33}$$

A similar analysis of axially rotating flows has been performed by Cousteix and Aupoix[23] using the LRR model. They obtained results similar to our Eqs. (13), (14), (31), and (32). However, due to the algebraic complexity of their model, they limited themselves to the case of small rotation retaining only terms linear in $Rr_y$. This led them to obtain constant values of $S_{MUU}$ and $S_{MUW}$, and therefore the effect of turbulence suppression under strong rotation has been excluded. Neglecting quadratic terms in $Rr_y$ in the equation for Reynolds stress was not self-consistent, leading to a spurious result of the Reynolds stress growing with $Rr_y$, in contradiction with Eq. (31). Their expression for the angle $\theta$ is, however, consistent with the linear approximation to Eq. (33).

Summarizing, we shall emphasize two main advantages of the proposed model over that by Cousteix and Aupoix[23]:

1) The QETE model provides expressions for the components of eddy viscosity tensor valid more generally than in just local equilibrium situation.

2) When the local equilibrium assumption is made, the model is simple enough to be analyzed analytically. One of the results of such an analysis is the criterion for extinction of three-dimensional turbulence by rotation.

## IV. Case of Spanwise Rotation

In this case, Eqs. (11) and (12) suggest that Reynolds stresses are related to the mean rates of strain as follows:

$$-\overline{uv} = q\ell S_{MU} \frac{\partial U}{\partial y} \tag{34}$$

$$-\overline{vw} = q\ell S_{MW} \frac{\partial W}{\partial y} \tag{35}$$

where

$$S_{MU} = B_1^{-1/3}/[1 + 36A_1^2 Ro_z^{-1}(Ru + Ro_z^{-1})] \tag{36}$$

$$S_{MW} = \frac{B_1^{-1/3}[1 + 9A_1^2 Ro_z^{-1}(Ru + 4Ro_z^{-1})]}{1 + 45A_1^2 Ro_z^{-1}(Ru + Ro_z^{-1}) + 324A_1^4 Ro_z^{-2}(Ru + Ro_z^{-1})^2} \tag{37}$$

$$Ru \equiv \frac{\ell}{q} \frac{\partial U}{\partial y} \tag{38}$$

and $Ro_z$ is a local Rossby number

$$Ro_z \equiv q/\ell f_z \tag{39}$$

As one can see, Reynolds stresses are aligned with mean strains, and so the eddy viscosity is a diagonal tensor, but its elements are not equal. Another important feature of this tensor is its explicit dependence on the mean shear $\partial U/\partial y$.

Let us consider the case of a unidirectional flow in which $W = 0$ (this configuration is similar to that created in the experiments of Johnston et al.[18]). For future discussion, we shall need to rewrite the equations for $\overline{u^2}$, $\overline{v^2}$, and $\overline{uv}$ using Eqs. (11), (12), (38), and (39):

$$\overline{u^2} = \tfrac{1}{3}[1 - (6A_1/B_1)]q^2 - 6A_1 \overline{uv} Ru - 6A_1 Ro_z^{-1} \overline{uv} \tag{40}$$

$$\overline{v^2} = \tfrac{1}{3}[1 - (6A_1/B_1)]q^2 + 6A_1 Ro_z^{-1} \overline{uv} \tag{41}$$

$$-\overline{uv} = 3A_1 q\ell \left[\frac{\overline{v^2}}{q^2}(1 + Rr_z) - \frac{\overline{u^2}}{q^2} Rr_z - C_1\right] \frac{\partial U}{\partial y} \tag{42}$$

where

$$Rr_z \equiv \frac{Ro_z^{-1}}{Ru} = \frac{f_z}{\partial U/\partial y} \tag{43}$$

is a stability parameter for spanwise rotating flows; $Rr_z$ has a significance similar to $Rr_y$ [Eq. (27)] in flows with axial rotation. Equations (34) and (42) suggest another form for $S_{MU}$:

$$S_{MU} = 3A_1 \left[\frac{\overline{v^2}}{q^2}(1 + Rr_z) - \frac{\overline{u^2}}{q^2} Rr_z - C_1\right] \tag{44}$$

Equations (40–42) show that $\overline{uv}$ redistributes turbulence energy between $\overline{u^2}$ and $\overline{v^2}$ and, in turn, $\overline{u^2}$ and $\overline{v^2}$ feedback on $\overline{uv}$. This feedback is an important component of the effect of rotation, but it has not been given enough attention (a brief discussion of the issue can be found in Hunt and Joubert[24]). To better understand the interaction between $\overline{u^2}$, $\overline{v^2}$, and $\overline{uv}$, we shall invoke the assumption of local equilibrium, providing in this case

$$-\overline{uv} \frac{\partial U}{\partial y} = \frac{q^3}{B_1 \ell} \tag{45}$$

Then it can be shown that

$$S_{MU} = B_1^{-1/3}[1 - 36A_1^2 B_1^{-2/3} Rr_z(1 + Rr_z)] \tag{46}$$

which is similar to an expression derived by So[14]; $S_{MU}$ is nonnegative if

$$-1.18 \le Rr_z \le 0.18 \tag{47}$$

Equations (40) and (41) can be rewritten using Eq. (45), providing

$$\frac{\overline{u^2}}{q^2} = \frac{1}{3}\left(1 + \frac{12A_1}{B_1}\right) + \frac{6A_1}{B_1} Rr_z \tag{48}$$

$$\frac{\overline{v^2}}{q^2} = \frac{1}{3}\left(1 - \frac{6A_1}{B_1}\right) - \frac{6A_1}{B_1} Rr_z \tag{49}$$

One can see that as $Rr_z$ varies in the range (47), both $\overline{u^2}/q^2$ and $\overline{v^2}/q^2$ vary in the range [0.163, 0.615], with $\overline{u^2}/q^2$ increasing and $\overline{v^2}/q^2$ decreasing with increasing $Rr_z$. For positive $Rr_z$, $S_{MU}$, given by Eq. (44), decreases and becomes zero at $Rr_z = 0.18$. On the other hand, when $Rr_z$ decreases below zero, $S_{MU}$ will increase until $Rr_z$ reaches the value $-0.5$, and the energy is distributed equally between $\overline{u^2}$ and $\overline{v^2}$; at this point $S_{MU}$ attains its maximum and $\overline{u^2} = \overline{v^2} = 0.389q^2$. As $Rr_z$ decreases further, the contributions of $\overline{u^2}$ and $\overline{v^2}$ to $S_{MU}$ switch, causing $S_{MU}$ to decrease again until it becomes zero at $Rr_z = -1.18$. This behavior of $S_{MU}$ is the direct result of the feedback of $\overline{u^2}$ and $\overline{v^2}$ on $\overline{uv}$, as described by Eq. (42). In Fig. 1, $S_{MU}$, $\overline{u^2}/q^2$, and $\overline{v^2}/q^2$, given by Eqs. (46), (48), and (49) are presented as functions of $Rr_z$. One can see that $S_{MU}$ is symmetrical about the axis $Rr_z = -0.5$, and $\overline{u^2}/q^2$, $\overline{v^2}/q^2$ are mirror images of each other. Theoretical and experimental results obtained by various authors suggest that positive $Rr_z$ decreases $S_{MU}$ causing stabilization of the mean flow, whereas negative $Rr_z$ has an opposite effect destabilizing the flow. This picture is in agreement with the above results for $Rr_z \geq -0.5$. For stronger destabilizing rotation, when $Rr_z < -0.5$, according to our results the flow should pass through the stages of restabilization. At $Rr_z = -1.18$, $S_{MU}$ becomes zero, and vertical turbulent mixing is completely suppressed. Experimental verification of the flow restabilization at strong destabilizing rotation awaits such data.

Pouagare and Lakshminarayana[25] employed an algebraic Reynolds stress model based on an LRR-type model by Rodi[4] and $k$-$\epsilon$ equations to simulate experimental data of Johnston et al.[18] The quadratic term in $Rr_z$ has been dropped from their equation for $S_{MU}$, providing

$$S_{MU} = (q^3/4\ell\epsilon)(0.13 - 0.5\, Rr_z) \tag{50a}$$

Obviously, such a model cannot account for restabilization. Since dissipation is given by $\epsilon = q^3/(B_1\ell)$, so that Eq. (50a) can be rewritten as

$$S_{MU} = 0.54 - 2.08\, Rr_z \tag{50b}$$

Pouagare and Lakshminarayana[25] have found that much better agreement with the data of Johnston et al.[18] is achieved if the coefficient 0.54 is changed to 0.37.

The equation given by Pouagare and Lakshminarayana[25] provides a linear approximation for $S_{MU}$, valid for weak rotational effect $|Rr_z| \ll 1$. In the same approximation, QETE model provides

$$S_{MU} = 0.39 - 1.84\, Rr_z \tag{50c}$$

The first numerical coefficient in Eq. (50b), 0.54, which had to be reset (quite arbitrarily) by Pouagare and Lakshminarayana[25] to the value of 0.37, has a value of 0.39 in QETE model, where it has been derived from the basic empirical constants. One should also keep in mind that Eq. (46) is an approximation to Eq. (36), derived by assuming of local equilibrium, and therefore Eq. (36) should be used in more complicated situations. An equation similar to Eq. (36) has recently been employed by Warfield and Lakshminarayana,[26] but it still requires the same resetting of the coefficient from 0.54 to 0.37.

Cousteix and Aupoix[23] suggested a similar model for spanwise rotating flows using on LRR model and local equilibrium assumption. Their final equations are similar to our Eqs. (46), (48), and (49), and the term quadratic in $Rr_z$ is retained in the expression for $\overline{uv}$, thus reflecting the effect of restabilization. The shortcomings of that model are again the necessity to reset the numerical coefficient in the expression for eddy viscosity for a duct flow without rotation, as Pouagare and Lakshminarayana[25] did, and the use of local equilibrium approximation. As has already been emphasized, the proposed QETE model is free from such deficiencies.

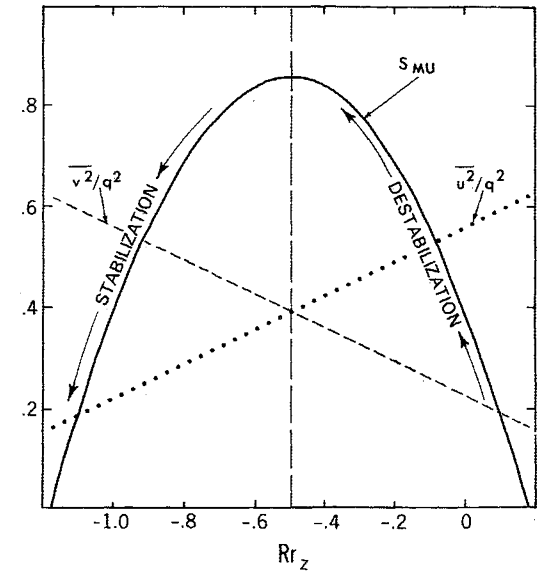


**Fig. 1 Variation of nondimensional eddy viscosity $S_{MU}$ and components of turbulence kinetic energy in spanwise rotating flow.**

It should be pointed out that if the basic empirical constants suggested by Gibson and Younis[16] are used instead, Eq. (50b) becomes

$$S_{MU} = 0.46 - 1.10\, Rr_z$$

which brings the first numerical coefficient closer to the adjusted value of Pouagare and Lakshminarayana.[25] On the other hand, this reduces the effect of rotation given by the second coefficient, which indicates that the re-evaluation of the constants by Gibson and Younis[16] provides only a limited improvement.

## V. Monin-Obukhov Similarity Theory for Rotation

Monin-Obukhov similarity theory was designed to describe the effects of density stratification on shear flows in the close vicinity of the surface where turbulent momentum and heat fluxes may be considered constant and local equilibrium approximation holds (Monin and Yaglom[27]). Hence, we develop a similarity theory for rotation for the constant-flux region. In this region, Reynolds stress remains constant and may be expressed as

$$-\overline{uv} = u_*^2 = q\ell\, S_{MU} \frac{\partial U}{\partial y} \tag{51}$$

where $u_*$ is the friction velocity. The local equilibrium approximation provides

$$-\overline{uv}\frac{\partial U}{\partial y} = u_*^2 \frac{\partial U}{\partial y} = \frac{q^3}{B_1 \ell} \tag{52}$$

where we assume $\ell = \kappa y$ near the wall. Assuming that in the constant flux layer mean shear can be expressed as

$$\frac{\partial U}{\partial y} = \frac{u_*}{\kappa y}\Phi_M(\zeta) \tag{53}$$

where $\Phi_M(\zeta)$ is a stability function and

$$\zeta \equiv \kappa y f_z / u_* \tag{54}$$

is a stability parameter, and using Eqs. (46) and (51-54), one can derive an equation for $\Phi_M(\zeta)$:

$$B_1^{-1/3}\Phi_M^2 - \frac{36A_1^2}{B_1}\zeta\Phi_M - B_1^{-1/3}\Phi_M^{2/3} - \frac{36A_1^2}{B_1}\zeta^2 = 0 \tag{55}$$

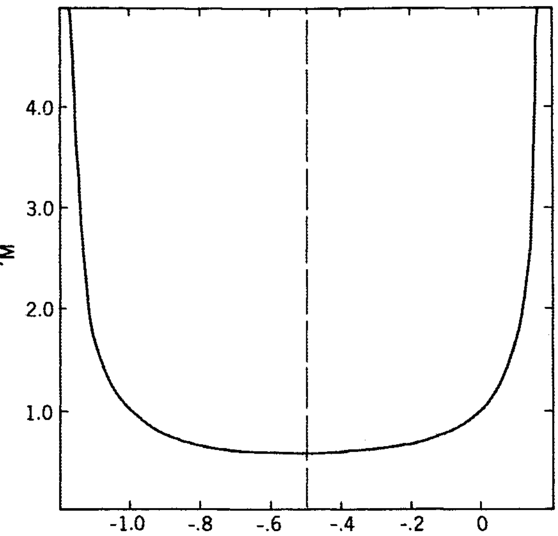


**Fig. 2 Universal stability function $\Phi_M(Rr_z)$ in the constant flux layer of spanwise rotating flow.**

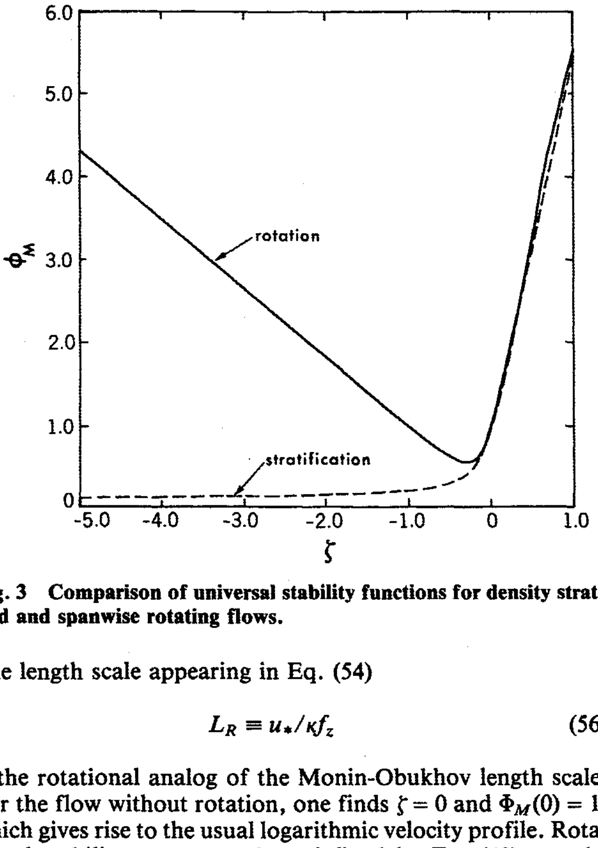


**Fig. 3 Comparison of universal stability functions for density stratified and spanwise rotating flows.**

The length scale appearing in Eq. (54)

$$L_R \equiv u_*/\kappa f_z \tag{56}$$

is the rotational analog of the Monin-Obukhov length scale. For the flow without rotation, one finds $\zeta = 0$ and $\Phi_M(0) = 1$, which gives rise to the usual logarithmic velocity profile. Rotational stability parameter $Rr_z$, defined by Eq. (43), can be written as

$$Rr_z = \zeta/\Phi_M(\zeta) \tag{57}$$

and equation for $\Phi_M$ becomes, in terms of $Rr_z$,

$$\Phi_M(Rr_z) = [1 - 36A_1^2B_1^{-2/3}Rr_z(1 + Rr_z)]^{-1/4} \tag{58}$$

Figure 2 shows the function $\Phi_M(Rr_z)$ in the range of $Rr_z$ allowed by Eq. (47). One can see that, similar to $S_{MU}$ in Fig. 1, $\Phi_M(Rr_z)$ is symmetrical with respect to the axis $Rr_z = -0.5$. Figure 3 compares the stability functions $\Phi_M(\zeta)$, calculated using the QETE model, for the cases of density stratification and spanwise rotation. For the former, $\zeta$ is defined as $\zeta = y/L$, $L$ being Monin-Obukhov length scale (see Mellor[28] for further details), and for the latter, Eq. (54) is employed. The curve for stratified flows is found to be in a good agreement with observations in the atmospheric boundary layers (Mellor[28], Businger et al.[29]). One can see that both curves are almost identical for $\zeta > 0$ where both flows are stabilized, and for $-0.28 \leq \zeta \leq 0$ where the flows are destabilized. Under stronger destabilizing conditions $\zeta < -0.28$, there is a substantial difference between the curves due to rotational restabilization.

If rotational effects are weak, $|\zeta| \ll 1$, and expansion in $\zeta$ is possible for the function $\Phi_M$, Eq. (55), yielding

$$\Phi_M(\zeta) = 1 + \beta\zeta \tag{59}$$

where

$$\beta = 27A_1^2B_1^{-2/3} = 3.51 \tag{60}$$

Substituting Eq. (59) in Eq. (53) and integrating between the surface roughness height or equivalent roughness of the laminar sublayer $y_0$, and some given height $y$, $y \gg y_0$, yields the log-linear velocity profile corrected for rotation:

$$\frac{U(y)}{u_*} = \frac{1}{\kappa}\left[\ln\left(\frac{y}{y_0}\right) + \beta\zeta\right] \tag{61}$$

The value of $\beta$ (3.51), obtained from the basic empirical constants, compares well with estimations by Watmuff et al.[30] of $\beta = 3$ to 5 in destabilized flows, and $\beta = 2$ to 4 in stabilized flows, and Koyama et al.[31] suggesting $\beta = 1$ to 4. Bradshaw[32] has found from experimental results that $\beta$ is about 4 for stabilizing and 2 for destabilizing rotation.

Similar analysis can be performed for the axial rotation case. Assuming that the surface stress is aligned with the $x$ direction and the friction velocity associated with this stress is $u_*$, the following equations can be derived:

$$\frac{\partial U}{\partial y} = \frac{u_*}{\kappa y}\Phi_{MU}(\zeta) \tag{62}$$

$$\frac{\partial W}{\partial y} = \frac{u_*}{\kappa y}\Phi_{MW}(\zeta) \tag{63}$$

where the stability parameter $\zeta$ is given by

$$\zeta \equiv \frac{\kappa y f_y}{u_*} = \frac{y}{L_R} \tag{64}$$

and where the axial rotation analog of the Monin-Obukhov length scale is

$$L_R \equiv \frac{u_*}{\kappa f_y} \tag{65}$$

which is similar to $L_R$ defined for spanwise rotation, Eq. (56). It can be shown that

$$\Phi_{MU}(\zeta) = 1 \tag{66}$$

$$\Phi_{MW}(\zeta) = \beta\zeta \tag{67}$$

where

$$\beta = 3A_1B_1^{-1/3} = 1.08 \tag{68}$$

It is interesting to note that unlike the case of spanwise rotation, Eq. (55), the effect of axial rotation is linear in $\zeta$. Also, the effect of axial rotation is weaker than its spanwise counterpart, which is readily seen via comparison of the numerical values of the coefficients $\beta$ in both cases, Eqs. (60) and (68).

Summarizing the results of this section, the following conclusions can be drawn:

1) There exist rotational analogs to the Monin-Obukhov length scale in both cases of spanwise and axial rotation in the constant flux region.

2) Rotational effects on the mean flow can be adequately captured by a similarity theory of the same kind as Monin-Obukhov theory for stratified flows. The universal similarity functions in this theory can be derived, as Mellor[28] did, by invoking the Reynolds stress equation.

3) The coefficient $\beta$, Eqs. (59), (60), (67), and (68) describing linear deviation from the logarithmic velocity profile, can be obtained from the basic assumptions of the Mellor family models without any additional semi-empiricism and is in good agreement with experimental results.

4) An important difference between the effects of spanwise rotation and stratification is the rotational restabilization that occurs under strong destabilizing rotation.

## VI. Conclusions

Rotational influence has often been taken into account in the past by somewhat arbitrary modifications of the length scale in a mixing length approach. The shortcoming of this approach is that the mean momentum equations alone are insufficient to describe the effects of rotation on turbulence structure. These effects are best studied by examining the Reynolds stress equations. For example, the effects of spanwise rotation on boundary-layer flows cannot be seen at the level of the mean momentum equations. On the other hand, Reynolds stress redistribution can be identified with the extra strains introduced by Bradshaw.[1]

The effects of rotation on neutral turbulent flows can be summarized as follows:

1) Coriolis terms retained in the Reynolds stress equations impart tensorial properties to eddy viscosity. Because of $f_z$ (spanwise rotation), the diagonal elements of the eddy viscosity tensor become unequal, and due to $f_y$ (axial rotation) this tensor gains off-diagonal elements.

2) Strong axial rotation in either direction tends to decrease the intensity of turbulence, ultimately leading to its extinction. Local equilibrium analysis provides critical values of the stability parameter indicating total suppression of turbulence.

3) Spanwise rotation may increase or decrease the intensity of turbulence, depending on the direction of rotation (or sign of the mean shear, which is equivalent). However, under very strong destabilizing rotation, destabilization is replaced by restabilization. The maximum in turbulent mixing occurs when turbulent energy is distributed equally among its components. The local equilibrium analysis again provides critical values of the stability parameter indicating extinction of turbulence.

4) Although effects of rotation are in certain respects similar to the effects of stratification, there is also an inherent difference between the two. Stratification directly affects the energy equation, causing extinction of turbulence in the stable case or totally overwhelming mechanical production and leading to the state of free convection in the unstable case. This is reflected by the value of the flux Richardson number. Rotation redistributes turbulence energy, but it does not enter energy equation explicitly. As a consequence, the energy equation cannot be used for introducing rotational stability parameters. Rotation alters the state of isotropy of turbulence. Our model suggests that maximum turbulent mixing takes place in the state of approximate isotropy. On the other hand, when turbulence gains too strong an anisotropy, three-dimensional turbulence ceases.

5) Monin-Obukhov-type similarity theory can be derived to describe the features of the mean flow with rotation in the region near the surface where the turbulent fluxes are constant.

6) The QETE model offers a self-consistent framework to include the effects of rotation. It possesses two prognostic equations for $q^2$ and $q^2\ell$ and, therefore, is not limited by the assumption of local equilibrium. It generates simple analytical relations that provide important insight into the structure of rotational turbulence. It does not require a posteriori calibration of its constants, and its results extend and generalize results obtained through the use of LRR-type closure models. The model also offers a self-consistent framework for inclusion combined effects of rotation, stratification (Galperin et al.[10] and Kantha et al.[11]), and streamline curvature.

7) Finally, it will be relevant to discuss the results of a recent paper by Speziale[33] who asserts that "all existing second-order closures yield spurious physical results for . . . test problem of rotating channel flow." Speziale[33] considers the case of very strong rotation $f \to \infty$, and he suggests that in this limit not one of the existing second-moment closure models accounts for the two-dimensionalization of turbulence and for the re-organization of the flow in accordance with the Taylor-Proudman theorem.

Our point of view is that since three-dimensional turbulence is significantly different from two-dimensional turbulence, no standard second-moment closure model can be made universal enough to be applicable in both cases. The existing closure models are designed to simulate a fully developed three-dimensional turbulence, with a well-defined inertial range. The limits of validity of these models can be expressed by inequalities involving various stability parameters such as in Eqs. (30) and (47) of the present paper. It was shown by Mellor and Yamada[8] that for $\overline{u_{\min}^2}/q^2 \leq 0.12$, $\overline{u_{\min}^2}$ being the minimum among the components of turbulence energy, Rotta hypothesis, the central element of the second-moment closure models, becomes invalid. Since two-dimensionalization of turbulence implies significant anisotropy, it is clear that it cannot be described by such models. However, the fact that these models break down for highly anisotropic turbulence (Speciale's $f \to \infty$ limit is one such case) by no means implies that they cannot be used for moderate rotations where the degree of anisotropy is small, as indeed is the case with turbomachinery. Besides, as is clear from the results in this paper, three-dimensional turbulence is often extinguished well before significant anisotropy is produced by rotational effects and well before Rotta hypothesis is violated.

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